

On the degree-1 Abel map for nodal curves

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Abstract

Let C be a nodal curve and L be an invertible sheaf on C . Let $\alpha_L : C \dashrightarrow J_C$ be the degree-1 rational Abel map, which takes a smooth point $Q \in C$ to $[m_Q \otimes L]$ in the Jacobian of C . In this work we extend α_L to a morphism $\bar{\alpha}_L : C \rightarrow \bar{J}_E^P$ taking values on Esteves' compactified Jacobian for any given polarization E . The maps $\bar{\alpha}_L$ are limits of Abel maps of smooth curves of the type α_L .

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1 Introduction

Let C be a nodal curve defined over an algebraically closed field K . Let J_C be the Jacobian of C and L be an invertible sheaf on C . The aim of this article is to construct a resolution of the rational Abel map

$$\begin{array}{ccc} \alpha_L : C & \dashrightarrow & J_C \\ Q & \mapsto & [m_Q \otimes L], \end{array}$$

where m_Q is the ideal sheaf of the point Q . If $Q \in C$ is smooth, then $\alpha_L(Q)$ is well-defined but if $Q \in C$ is a node, then $\alpha_L(Q)$ is not defined because m_Q is noninvertible.

In order to solve that map, that is, to extend the Abel map on the whole C , we must enlarge the target space of α_L , which leads us to the problem of how to find a good compactification for the Jacobian. The problem of the compactification goes back at least to Igusa [I]. Later on D'Souza, following a suggestion of Mumford and Mayer, obtained in his thesis [DS] a compactification of relative Jacobian of a family of irreducible curves with nodes and cusps as singularities under somewhat restrictive hypothesis. One year later, Altman and Kleiman [AK80] gave a good solution for the case of families of geometrically integral curves. Their relative compactification parametrizes torsion-free and rank-1 sheaves on the fibers, and it admits a universal sheaf after an étale base change.

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For reducible nodal curves Oda-Seshadri [OS] and Seshadri [S] produced some compactifications. But these compactifications are not applicable to families of reduced curves. In her thesis [C], Caporaso constructed a compactification for the relative generalized Jacobians of families of stable curves. One year later, Pandharipande [PP] produced in his thesis an equivalent construction, valid for higher ranks as well. At nearly the same time, Simpson [Si] constructed moduli spaces of coherent sheaves over any family of projective varieties. The compactifications by Caporaso, Pandharipande and Simpson are not fine moduli spaces, and thus do not carry a Poincaré sheaf.

At last, Esteves considered in [E01] the algebraic space constructed by Altman and Kleiman [AK80], parametrizing torsion-free rank-1 simple sheaves on the fibers of a family of curves and he showed that this space is universally closed over the base, and consequently one can regard it as a compactification of the relative Jacobian. This compactification is a fine moduli space, and hence it does admit a Poincaré sheaf after an étale base change. In this work we consider Abel maps into Esteves' compactification.

We recall that the map α_L above is the case $d = 1$ of the more general rational degree- d Abel map α_L^d defined, for a positive integer d and a line bundle L , in the following way:

$$\begin{aligned} \alpha_L^d : \quad C^d & \dashrightarrow J_C \\ (Q_1, \dots, Q_d) & \mapsto [m_{Q_1} \otimes \dots \otimes m_{Q_d} \otimes L]. \end{aligned}$$

When C is smooth, α_L^d is a morphism and a well-know result of Abel says that the fibers of Abel map of degree d are projectivized complete linear series (up to the natural action of the d -th symmetric group). So, when C is smooth, all the possible embeddings of C in projective spaces are known once we know its Abel maps.

Some particular cases of the Abel maps have been solved. Altman and Kleiman in [AK80] considered the problem for integral curves. For reducible curves, the problem of completing the Abel maps is open with a few exceptions: Caporaso and Esteves in [CE] constructed degree-1 Abel maps for stable curves; Caporaso, Coelho and Esteves in [CCoE] constructed a degree-1 Abel maps for Gorenstein curves; Coelho and Pacini in [CoP] constructed Abel maps of any degree for curves of compact type; Pacini in [P1] and [P2] constructed a degree-2 Abel maps for nodal curves and finally Abreu, Coelho and Pacini in [ACoP] constructed degree- d Abel maps for nodal curves with two components. In all cases the authors have been used a specific polarization and a particular L .

The general strategy to solve the Abel maps is to resort to families of curves. More precisely, let C be a nodal curve of genus g and $f : \mathcal{C} \rightarrow B$ be a regular local smoothing of C , i.e., a family of curves where \mathcal{C} is smooth and B is the spectrum of a DVR (discrete valuation ring) with residue field k and quotient field K , and such that f has special fiber isomorphic to C and smooth generic fiber \mathcal{C}_K . Let $\sigma : B \rightarrow \mathcal{C}$ be a section of f through the B -smooth locus of \mathcal{C} . Let \mathcal{E} be a polarization on \mathcal{C} , i. e., a vector bundle on \mathcal{C} such that $\text{rk}(\mathcal{E})$ divides $\deg(\mathcal{E})$. Let L be a line bundle on C of degree $g - \mu(\mathcal{E})$, and consider a deformation \mathcal{L} of L , i. e., an invertible sheaf \mathcal{L} on \mathcal{C} such that $\mathcal{L}|_C = L$.

Consider the scheme $\overline{\mathcal{J}}_{\mathcal{E}}^{\sigma}$ constructed in [E01], parametrizing torsion-free rank-1 sheaves I of degree $(g - 1 - \mu(\mathcal{E}))$ on \mathcal{C}/B that are σ -quasistable with respect to \mathcal{E} . This means that I satisfies certain numerical conditions depend-

ing on the degree of I in each component of C . We recall that $\overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$ is a proper B -scheme.

Given \mathcal{L} and \mathcal{E} as above, we have a rational map

$$\alpha_{\mathcal{L},\mathcal{E}} : \mathcal{C} \dashrightarrow \overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$$

defined over \mathcal{C}_K which takes $Q \in \mathcal{C}_K$ to $\alpha_{\mathcal{L},\mathcal{E}}(Q) = [m_Q \otimes \mathcal{L}|_{\mathcal{C}_K}]$. Our aim is to extend this map to the whole \mathcal{C} . Since $\overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$ is a fine moduli space, to extend the map $\alpha_{\mathcal{L},\mathcal{E}}$ to the whole \mathcal{C} it is enough to give a relatively torsion-free rank-1 σ -quasistable sheaf \mathcal{M} on the family

$$p_1 : \mathcal{C} \times_B \mathcal{C} \rightarrow \mathcal{C}$$

given by the projection map p_1 onto the first factor. The moduli map induced by \mathcal{M} is given by its restriction over the fibers of the family p_1 .

As we will see in Theorem 20, for any invertible sheaf \mathcal{L} and for any polarization \mathcal{E} we can show that $\alpha_{\mathcal{L},\mathcal{E}} : \mathcal{C} \dashrightarrow \overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$ is already a morphism. More precisely, let $\phi : \widehat{\mathcal{C}}^2 \rightarrow \mathcal{C}^2$ be a good partial desingularization. We will be able to construct an invertible sheaf $\widetilde{\mathcal{F}}$ over $\widehat{\mathcal{C}}^2$, such that $\phi_* \widetilde{\mathcal{F}}$ is a relatively torsion-free, rank-1, $\widetilde{\sigma}$ -quasistable sheaf over \mathcal{C}^2 , where $\widetilde{\sigma} : \mathcal{C} \rightarrow \mathcal{C}^2$ is the section of the projection $\mathcal{C}^2 \rightarrow \mathcal{C}$ onto the second factor induced by the section $\sigma : B \rightarrow \mathcal{C}$. This sheaf induces a morphism

$$\overline{\alpha}_{\mathcal{L},\mathcal{E}} : \mathcal{C} \rightarrow \overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$$

which takes $Q \in \mathcal{C}$ to

$$\phi_* \widetilde{\mathcal{F}} \Big|_{p_1^{-1}(Q)}$$

restricting to $\alpha_{\mathcal{L},\mathcal{E}}$ over the smooth locus of $f : \mathcal{C} \rightarrow B$.

We recall that in [CCoE], [CE] and [Co] the authors have been used a particular polarization and the particular $L = \mathcal{O}_C(P)$. Notice that we use the approach used by Caporaso and Esteves in [CE] where the obstruction to extend the Abel map is overcome by using a special type of invertible sheaves, named twistors. Rocha, in his thesis [R] constructed degree-1 and degree-0 Abel maps avoiding the use of twistors, putting Simpson's compactified Jacobians as the target of Abel maps.

In short, here is a summary of this article. In Section 2 we review the technical background, in particular the concepts of P -quasistability and σ -quasistability. In Sections 3 and 4 we define twistors and twistor difference. In Section 5 we construct the sheaf \mathcal{M} on $\mathcal{C} \times_B \mathcal{C}/\mathcal{C}$ which solves the first Abel map.

2 Technical background

Let K be an algebraically closed field. A *curve* is a connected, projective and reduced scheme of dimension 1 over K . A *subcurve* Y of a curve C is a reduced subscheme of pure dimension 1, or equivalently, a reduced union of irreducible components of C . A *node* of a curve C is a singular point N of C such that $\widehat{\mathcal{O}}_{C,N} = k[[x,y]]/(xy)$. A node N of C is called a *separating node* if there is a subcurve Y of C such that $Y \cap Y' = \{N\}$, where $Y' = \overline{C \setminus Y}$. A node N of C is called *reducible* if there are C_i and C_j , distinct irreducible components of

C , such that $N \in C_i \cap C_j$, otherwise, N is called an *irreducible node*. A *nodal curve* is a curve with only nodal singularities. We denote by C^{sing} and C^{sm} respectively the singular and smooth locus of C .

If $Y, Z \subseteq C$ are subcurves, we let $Y \wedge Z$ denote the maximum subcurve of C contained in $Y \cap Z$; we let $\overline{Z - Y}$ denote the minimum subcurve containing $Z \setminus Y$. If $Y, Z \subseteq C$ are subcurves such that $\dim(Y \cap Z) = 0$, define δ_{YZ} as the number of nodes in $Y \cap Z$. In addition, if $Y = C_i$ and $Z = C_j$ are irreducible components of a curve C we use $\delta_{i,j}$ to denote $\delta_{C_i C_j}$.

A *chain of rational curves* is a curve whose components are smooth and rational and can be ordered, E_1, \dots, E_n , in such a way that $\#(E_i \cap E_{i+1}) = 1$ for $i = 1, \dots, n-1$ and $E_i \cap E_j = \emptyset$ if $|i - j| > 1$. If n is the number of components, we say that the chain has *length* n . The components E_1 and E_n are called the *extreme curves* of the chain.

Let \mathcal{N} be a collection of nodes of C , and $\eta : \mathcal{N} \rightarrow \mathbb{N}$ a function. Denote by $\tilde{C}_{\mathcal{N}}$ the partial normalization of C at \mathcal{N} . For each $P \in \mathcal{N}$, let E_P be a chain of rational curves of length $\eta(P)$. Let C_{η} denote the curve obtained as the union of $\tilde{C}_{\mathcal{N}}$ and the E_P for $P \in \mathcal{N}$ in the following way: each chain E_P intersects no other chain, but intersects $\tilde{C}_{\mathcal{N}}$ transversally at two points, the branches over P on $\tilde{C}_{\mathcal{N}}$ on one hand, and nonsingular points on each of the two extreme curves of E_P on the other hand. There is a natural map $\mu_{\eta} : C_{\eta} \rightarrow C$ collapsing each chain E_P to a point, whose restriction to $\tilde{C}_{\mathcal{N}}$ is the partial normalization map. The curve C_{η} and the map μ_{η} are well-defined up to C -isomorphism. A rational curve in any of the chain E_P is called μ_{η} -*exceptional curve*. We call the curve C_{η} (or the map μ_{η}), a *semistable modification* of C . If $\eta(P) = 1$ for each $P \in \mathcal{N}$, then C_{η} (or the map μ_{η}) is called a *quasistable modification* of C .

There are two special cases of the above construction that will be useful for us. First, if $\mathcal{N} = \{R\}$ and η takes R to 1, let $C_R := C_{\eta}$ and $\mu_R := \mu_{\eta}$. Second, if $\mathcal{N} = \mathcal{N}(C)$, where $\mathcal{N}(C)$ is the collection of reducible nodes of C , and $\mu : \mathcal{N}(C) \rightarrow \mathbb{N}$ is the constant function with value m , let $C(m) := C_{\eta}$ and $\mu(m) := \mu_{\eta}$. Set $C(0) := C$ and $\mu(0) := \text{id}_C$.

Given a map of curves $\phi : C' \rightarrow C$, we say that an irreducible component of C' is ϕ -*exceptional* if it is a smooth rational curve and is contracted by the map.

Let I be a coherent sheaf on a curve C . We say that I is *torsion-free* if its associated points are generic points of C . We say that I is *of rank 1* if I is invertible on a dense open subset of C . If I is a rank-1 torsion-free sheaf, we call $\deg(I) := \chi(I) - \chi(\mathcal{O}_C)$ the *degree* of I and we define

$$I_Y := \frac{I|_Y}{\mathcal{T}(I|_Y)},$$

where $\mathcal{T}(I|_Y)$ is the torsion subsheaf of $I|_Y$. Note that I_Y is torsion-free. A sheaf I is said to be *simple* if $\text{End}(I) = k$, or equivalently, if I is not decomposable [Co, Lemma 1.1.5, p. 11].

Let E be a vector bundle on a curve C . The *slope* of E is the number

$$\mu(E) := \frac{\deg(E)}{\text{rk}(E)}.$$

A *polarization* on a curve C , in the sense of Esteves [E01], is a vector bundle E on C whose slope is an integer, that is, such that $\text{rk}(E)$ divides $\deg(E)$.

A torsion-free rank-1 sheaf I on a curve C is *semistable* with respect to E if $\chi(E \otimes I) = 0$ and $\chi(E \otimes I_Y) \geq 0$, for all proper subcurves Y of C . If P is a nonsingular point of C , we say that a torsion-free rank-1 sheaf I is *P -quasistable* with respect to E if I is semistable and in addition $\chi(E \otimes I_Y) > 0$ for every proper subcurve Y of C containing P . Let

$$\beta_I(Y) := \chi(I_Y) + \frac{\deg_Y(E)}{\mathrm{rk}(E)}.$$

As $\chi(E \otimes I_Y) = \mathrm{rk}(E)\chi(I_Y) + \deg_Y(E)$, the sheaf I is P -quasistable if and only if $\beta_I(Y) \geq 0$ for every proper subcurve Y of X and $\beta_I(Y) > 0$ for every subcurve Y such that $P \in Y$.

A *family of curves* is a proper and flat morphism $f : \mathcal{C} \rightarrow B$ whose fibers are curves. If $b \in B$, we denote by $\mathcal{C}_b := f^{-1}(b)$ its fiber. The family $f : \mathcal{C} \rightarrow B$ is called *local* if $B = \mathrm{Spec}(K[[t]])$, *regular* if \mathcal{C} is regular and *pointed* if it is endowed with a section $\sigma : B \rightarrow \mathcal{C}$ through the smooth locus of f . A *smoothing* of a curve C is a regular local family $f : \mathcal{C} \rightarrow B$ with special fiber C . A *sheaf* on \mathcal{C}/B is a B -flat coherent sheaf on \mathcal{C} . Given a pointed smoothing $f : \mathcal{C} \rightarrow B$ of a curve C with section $\sigma : B \rightarrow \mathcal{C}$, we define $P := \sigma(0)$. If $f : \mathcal{C} \rightarrow B$ is a family of curves, we denote by \mathcal{C}^2 the product $\mathcal{C} \times_B \mathcal{C}$ and by \mathcal{C}_T the product $\mathcal{C} \times_B T$ where T is a B -scheme.

Let $f : \mathcal{C} \rightarrow B$ a family of curves and let \mathcal{I} be a sheaf on \mathcal{C}/B . The sheaf \mathcal{I} is called *torsion-free* (resp. *rank-1* and *simple*) on \mathcal{C}/B if, for each $b \in B$, the restriction $\mathcal{I}(b)$ is torsion-free (resp. rank-1 and simple).

Let $f : \mathcal{C} \rightarrow B$ be a smoothing of C . The *relative compactified Jacobian functor* of the family \mathcal{C}/B is the contravariant functor

$$\overline{\mathcal{J}}_{\mathcal{C}/B} : (\mathrm{Sch}/B)^\circ \longrightarrow (\mathrm{Sets})$$

that associates to each B -scheme T the set of simple, torsion-free, rank-1 sheaves on \mathcal{C}_T/T modulo equivalence, where we say that two sheaves \mathcal{F}_1 and \mathcal{F}_2 on \mathcal{C}_T/T are equivalent if there exists an invertible sheaf \mathcal{G} on T such that $\mathcal{F}_1 \simeq \mathcal{F}_2 \otimes p_2^*(\mathcal{G})$, with $p_2 : \mathcal{C}_T \rightarrow T$ being the projection onto the second factor.

Since $f : \mathcal{C} \rightarrow B$ is a family of curves, the functor $\overline{\mathcal{J}}_{\mathcal{C}/B}$ is represented by an *algebraic space* $\overline{\mathcal{J}}_{\mathcal{C}/B}$ [AK80, Theorem 7.4, p. 99]. Esteves showed that, after a suitable étale base change to obtain enough sections [E01, Lemma 18, p. 3061], $\overline{\mathcal{J}}_{\mathcal{C}/B}$ becomes a *scheme* [E01, Theorem B, p. 3048], consequently it is a fine moduli space. But the space $\overline{\mathcal{J}}_{\mathcal{C}/B}$ is not proper.

Let $f : \mathcal{C} \rightarrow B$ be a family of curves and consider a vector bundle \mathcal{E} on \mathcal{C} . The *relative degree* of \mathcal{E}/B is $\deg(\mathcal{E}/B) := \deg_{\mathcal{C}(b)}(\det(\mathcal{E}|_{\mathcal{C}(b)}))$, where $\mathcal{C}(b)$ is the fiber of \mathcal{C}/B over any $b \in B$. The quotient

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E}/B)}{\mathrm{rk}(\mathcal{E})}$$

is called the *slope* of \mathcal{E} . We say \mathcal{E} is a *polarization* if $\mathrm{rk}(\mathcal{E})$ divides $\deg(\mathcal{E}/B)$ (or equivalently $\mu(\mathcal{E})$ is an integer). Let $\sigma : B \rightarrow \mathcal{C}$ a section of f through the smooth locus of \mathcal{C} . A torsion-free rank-1 sheaf \mathcal{I} on \mathcal{C}/B is σ -quasistable with respect to \mathcal{E} if $\mathcal{I}(b)$ is $\sigma(b)$ -quasistable with respect to $\mathcal{E}(b)$ for every geometric point b of B . According to [Co, Lemma 1.3.5, p. 19], if \mathcal{I} is σ -quasistable, then \mathcal{I} is simple on \mathcal{C}/B .

Denote by $\overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$ the subspace of $\overline{\mathcal{T}}_{\mathcal{C}/B}$ parametrizing the torsion-free rank-1 sheaves \mathcal{I} on \mathcal{C}/B that are σ -quasistable with respect to \mathcal{E} . Esteves showed that $\overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$ is proper over B [E01, Theorem A, p. 3047].

Since that $\overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$ is a fine moduli space, to give a morphism $\alpha : \mathcal{C} \rightarrow \overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$ is equivalent to give a simple, torsion-free, rank-1 sheaf \mathcal{M} on $\mathcal{C} \times_B \mathcal{C}/\mathcal{C}$ which is σ -quasistable on the fibers. Given \mathcal{L} an invertible sheaf on \mathcal{C}/B and \mathcal{E} a polarization on \mathcal{C} , we have a rational map

$$\alpha_{\mathcal{L}} : \mathcal{C} \dashrightarrow \overline{\mathcal{T}}_{\mathcal{E}}^{\sigma},$$

where $\alpha_{\mathcal{L}}(Q) = [m_Q \otimes \mathcal{L}_K]$ for $Q \in \mathcal{C}_K$, \mathcal{C}_K is the generic fiber of $f : \mathcal{C} \rightarrow B$ and $\mathcal{L}_K = \mathcal{L}|_{\mathcal{C}_K}$. So, to extend this map it suffices to give a simple, torsion-free, rank-1 sheaf \mathcal{M} on $\mathcal{C} \times_B \mathcal{C}/\mathcal{C}$ which is σ -quasistable on the fibers, such that for $Q \in \mathcal{C}_K$,

$$\mathcal{M}|_{\mathcal{C} \times_B \{Q\}} = m_Q \otimes \mathcal{L}|_{\mathcal{C}_K}.$$

Given a smoothing \mathcal{C}/B of C , a *twister* of \mathcal{C}/B is a line bundle of degree-0 on C of the form $\mathcal{O}_{\mathcal{C}}(Z)|_C$, where Z is a Cartier divisor of \mathcal{C} supported in C , so a formal sum of components of C of the type $\sum a_i C_i$ where $a_i \in \mathbb{Z}$. We set $\mathcal{O}_C(Z) := \mathcal{O}_{\mathcal{C}}(Z)|_C$.

3 Nodal curves and twistors

Let C be a connected, nodal curve with components C_1, \dots, C_p . Let $f : \mathcal{C} \rightarrow B$ be a regular local smoothing of C . Let $\sigma : B \rightarrow \mathcal{C}$ be a section of f through the B -smooth locus of \mathcal{C} . Let \mathcal{E} be a polarization on \mathcal{C} . Let L be a line bundle on C , and consider a deformation \mathcal{L} of L , i. e., an invertible sheaf \mathcal{L} on \mathcal{C}/B such that $\mathcal{L}|_C = L$.

Let $\phi : \widetilde{\mathcal{C}}^2 \rightarrow \mathcal{C}^2$ be a desingularization of \mathcal{C}^2 . Let $\Delta \subset \mathcal{C}^2$ be the diagonal subscheme and let $\widetilde{\Delta}$ be the strict transform of Δ (via ϕ). Denoting by $p_2 : \mathcal{C}^2 \rightarrow \mathcal{C}$ the second projection, we obtain a family of curves

$$\widetilde{\mathcal{C}}^2 \xrightarrow{\phi} \mathcal{C}^2 \xrightarrow{p_2} \mathcal{C}$$

Consider the following sheaf over the family $\widetilde{\mathcal{C}}^2/\mathcal{C}$

$$\widetilde{\mathcal{M}} := \mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}}^2} \otimes (p_2\phi)^* \mathcal{L} \otimes \widetilde{\mathcal{T}}, \quad (1)$$

where $\widetilde{\mathcal{T}}$ is an invertible sheaf.

We want to find a desingularization $\phi : \widetilde{\mathcal{C}}^2 \rightarrow \mathcal{C}^2$ so that $\phi_* \widetilde{\mathcal{M}}$ is a relatively torsion-free, rank-1, σ -quasistable sheaf on $\mathcal{C}^2/\mathcal{C}$.

The map

$$\overline{\alpha}_{\mathcal{L}, \mathcal{E}} : \mathcal{C} \rightarrow \overline{\mathcal{T}}_{\mathcal{E}}^{\sigma}$$

induced by $\phi_* \widetilde{\mathcal{M}}$ coincides with $\alpha_{\mathcal{L}, \mathcal{E}}$ over the generic fiber of f

3.1 Good partial desingularizations

By [CoEP, Section 3.1], the 3-fold \mathcal{C}^2 is singular exactly at the points (R, S) where R, S are nodes of C , not necessarily distinct, i. e.,

$$\text{Sing}(\mathcal{C}^2) = \{(R, S) : R, S \in C^{\text{sing}}\}.$$

In fact, \mathcal{C}^2 has a quadratic isolated singularity at (R, S) . This singularity can be resolved by blowing up \mathcal{C}^2 at (R, S) , at the cost of replacing the point by a quadric surface. However, for our purposes, we choose a desingularization that replaces each point (R, S) by a unique smooth rational curve. Thus, firstly we need a convenient desingularization of \mathcal{C}^2 . For more details on the subject, we refer the reader to [CoEP, Sections 3 and 4].

Let $\phi : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}^2$ be a *good partial resolution of singularities* of \mathcal{C}^2 as introduced by [CoEP, Section 4, p.25], that is, $\phi : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}^2$ is a sequence of blowups, starting by the blowup along the diagonal subscheme of \mathcal{C}^2 and then blowing up all strict transform of products $Y \times Z$ of irreducible components Y and Z of C with $Y \neq Z$. According to [CoEP, Section 4.1, p. 27], $\widetilde{\mathcal{C}^2}$ is nonsingular away from the points over the pairs (R, S) of distinct nodes R, S of C , so the strict transform, via ϕ , of any product $Y \times Z$ is a Cartier divisor in $\widetilde{\mathcal{C}^2}$.

In this context the family of curves

$$\widetilde{\mathcal{C}^2} \xrightarrow{\phi} \mathcal{C}^2 \xrightarrow{p_2} \mathcal{C}$$

looks locally over a node $R \in C$ like the below diagram

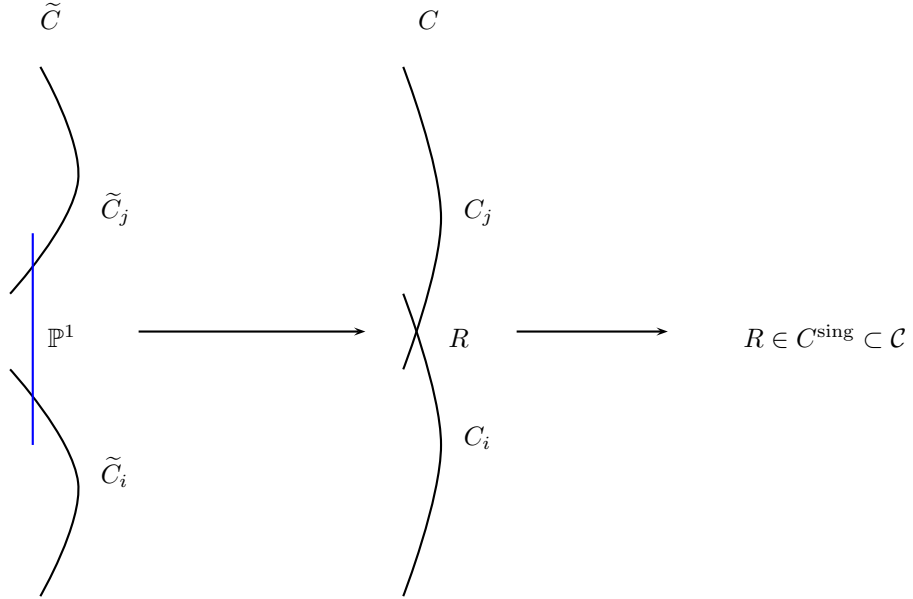


Figure 1: Fiber over $R \in C^{\text{sing}} \subset C$ of the family $p_2 \circ \phi$.

Let C be a nodal curve. We define C_R as the curve obtained from C by replacing the node R by a smooth rational curve, and $C(1)$ as the curve obtained from C by replacing each reducible node of C by a smooth rational curve. Let

$\mathcal{I}_{\Delta|\mathcal{C}^2}$ be the ideal sheaf of $\Delta \subset \mathcal{C}^2$ and let

$$\mathbb{P}_{\mathcal{C}^2}(\mathcal{I}_{\Delta|\mathcal{C}^2}) := \text{Proj}_{\mathcal{C}^2}(\mathcal{S}(\mathcal{I}_{\Delta|\mathcal{C}^2})),$$

where $\mathcal{S}(\mathcal{I}_{\Delta|\mathcal{C}^2})$ is the sheaf of symmetric algebras of $\mathcal{I}_{\Delta|\mathcal{C}^2}$.

The next three propositions summarize the properties of the good partial desingularizations we are looking for

Proposition 1 *Let $\phi : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}^2$ be the blowup of \mathcal{C}^2 along Δ . Let $\rho_i := p_i \phi$, where $p_i : \mathcal{C}^2 \rightarrow \mathcal{C}$ is the projection onto the i -th factor for $i = 1, 2$. Let $R \in \mathcal{C}$. For $i = 1, 2$, let $X_i := \rho_i^{-1}(R)$ and denote by $\mu_i : X_i \rightarrow \mathcal{C}$ the restriction of ρ_{3-i} to X_i . Then the following statements hold:*

- 1) $\widetilde{\mathcal{C}^2}$ is \mathcal{C}^2 -isomorphic to $\mathbb{P}_{\mathcal{C}^2}(\mathcal{I}_{\Delta|\mathcal{C}^2})$.
- 2) ρ_i is flat for $i = 1, 2$.
- 3) If R is not a node of \mathcal{C} , then μ_i is an isomorphism for $i = 1, 2$.
- 4) If R is a node of \mathcal{C} , then X_i is \mathcal{C} -isomorphic to \mathcal{C}_R and $\widetilde{\mathcal{C}^2}$ is regular along the rational component of X_i contracted by μ_i for $i = 1, 2$.

Proof. [CoEP, Proposition 2.2, p.14] ■

We have the following diagram

$$\begin{array}{ccccc}
 & & \rho_2 & & \\
 & \nearrow \phi & & \searrow p_2 & \\
 \widetilde{\mathcal{C}^2} & \xrightarrow{\quad} & \mathcal{C}^2 & \xrightarrow{\quad} & \mathcal{C} \\
 & \searrow \rho_1 & \downarrow p_1 & & \downarrow f \\
 & & \mathcal{C} & \xrightarrow{\quad} & B
 \end{array}$$

where p_1 and p_2 are, respectively, the projections onto the first and second factors.

Proposition 2 *Let R and S be reducible nodes of \mathcal{C} . Assume $R \in C_i \cap C_j$ and $S \in C_k \cap C_l$, for integers i, j, k, l with $i \neq j$ and $k \neq l$. If $R = S$, assume $i = k$ and $j = l$. Let $\phi : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}^2$ denote the blowup of \mathcal{C}^2 along $C_i \times C_l$, or along the diagonal if $R = S$. Put $E := \phi^{-1}(R, S)$. Then the following statements hold:*

- 1) E is a smooth rational curve and $\widetilde{\mathcal{C}^2}$ is regular in a neighborhood of E .
- 2) The strict transforms of $C_i \times C_l$ and $C_j \times C_k$ contain E , while those of $C_i \times C_k$ and $C_j \times C_l$ intersect E transversally at a unique point, distinct for each transform.
- 3) If $R = S$, the strict transform of the diagonal contains E .
- 4) The composition $\widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}$ of ϕ with the projection of \mathcal{C}^2 onto any of its factors is flat.

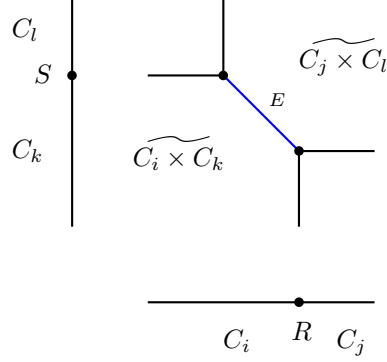


Figure 2: Blow up along $C_i \times C_k$. The strict transforms of $C_i \times C_k$ and $C_j \times C_l$ contains E .

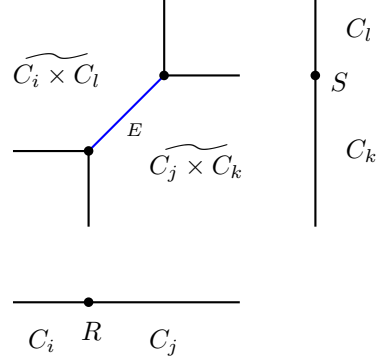


Figure 3: Blow up along $C_i \times C_l$. The strict transforms of $C_i \times C_l$ and $C_j \times C_k$ contains E .

Proof. [CoEP, Proposition 3.1, p.17] ■

The pictures 2 and 3 below illustrates the possible blowups along a product of irreducible subcurves of C .

We denote by $\mathcal{N}(C)$ the collection of reducible nodes of C and by $\mathcal{N}_{i,k}(C)$ the subset of $\mathcal{N}(C)^2$ containing every pair of nodes (R, S) such that $R \in C_i$ and $S \in C_k$. Let $\phi : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}^2$ be a good partial desingularization of \mathcal{C}^2 . Denote by $\mathcal{N}_{i,k}(\phi)$ the subset of $\mathcal{N}_{i,k}(C)$ formed by pairs (R, S) such that $\phi^{-1}(R, S) \subset \widetilde{C_i \times C_k}$.

Proposition 3 *Let $\phi : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}^2$ be a good partial desingularization. Let $\rho : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}$ denote its composition with the first projection $p_1 : \mathcal{C}^2 \rightarrow \mathcal{C}$. Let $R \in \mathcal{C}$ and $\widehat{C} := \rho^{-1}(R)$. Let $\mu : \widehat{C} \rightarrow \mathcal{C}$ be the restriction to \widehat{C} of ϕ composed with the second projection $p_2 : \mathcal{C}^2 \rightarrow \mathcal{C}$. Then, the following statements hold:*

- 1) ρ is flat.
- 2) $\widetilde{\mathcal{C}^2}$ is regular along each smooth rational curve of \widehat{C} contracted by μ .
- 3) If R is not a node of C , then μ is an isomorphism.
- 4) If R is an irreducible node of C , then \widehat{C} is C -isomorphic to C_R .
- 5) If R is a reducible node of C , then \widehat{C} is C -isomorphic to $C(1)$.

Furthermore, for each $i, k \in \{1, \dots, p\}$, let $D_{i,k}$ denote the strict transform to $\widetilde{\mathcal{C}^2}$ of $C_i \times C_k$. Consider the Cartier divisor

$$D = \sum_{i,k} w_{i,k} D_{i,k}$$

for given integers $w_{i,k}$. Then, if R is a reducible node of \widehat{C} , the restriction $\mathcal{O}_{\widetilde{\mathcal{C}^2}}(D)|_{\widehat{C}}$ is a twister of \widehat{C} . More specifically, for each $i = 1, \dots, p$, let \widehat{C}_i be

the strict transform to \widehat{C} of C_i via μ , and for each reducible node S of C , let $E_S := \mu^{-1}(S)$. Then

$$\mathcal{O}_{\widehat{\mathcal{C}^2}}(D)|_{\widehat{C}} = \mathcal{O}_{\widehat{C}} \left(\sum_{k=1}^p a_k \widehat{C}_k + \sum_S b_S E_S \right), \quad (2)$$

where $a_k := \sum_{i,k} w_{i,k}$, the sum over the two i such that $R \in C_i$, and $b_S := \sum_{i,k} w_{i,k}$, the sum over two pairs (i,k) such that $(R, S) \in \mathcal{N}_{i,k}(\phi)$.

Proof. [CoEP, Proposition 4.4, p.27] ■

For instance, let $C = C_1 \cup C_2$ with $C_1 \cap C_2 = \{S\}$ and let $f : \mathcal{C} \rightarrow B$ be a regular smoothing of C . Suppose that $\phi : \widehat{\mathcal{C}^2} \rightarrow \mathcal{C}^2$ is a blowup of \mathcal{C}^2 and consider a Cartier divisor

$$D = w_{1,1}D_{1,1} + w_{1,2}D_{1,2} + w_{2,1}D_{2,1} + w_{2,2}D_{2,2}.$$

If ϕ is the blowup along $C_1 \times C_1$, then

$$\mathcal{O}_{\widehat{\mathcal{C}^2}}(D)|_{\widehat{C}} \simeq \mathcal{O}_{\widehat{C}} \left((w_{1,1} + w_{2,1})\widehat{C}_1 + (w_{1,2} + w_{2,2})\widehat{C}_2 + (w_{1,1} + w_{2,2})E_S \right).$$

If ϕ is the blowup along $C_1 \times C_2$, then

$$\mathcal{O}_{\widehat{\mathcal{C}^2}}(D)|_{\widehat{C}} \simeq \mathcal{O}_{\widehat{C}} \left((w_{1,1} + w_{2,1})\widehat{C}_1 + (w_{1,2} + w_{2,2})\widehat{C}_2 + (w_{1,2} + w_{2,1})E_S \right).$$

By definition, \widehat{C} contains a copy of each component C_i of C and contains an exceptional component glued at each node of C (see figure 4)

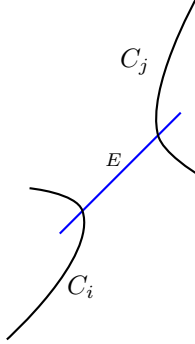


Figure 4: \widehat{C} locally around an ϕ -exceptional component E .

3.2 Admissible sheaves

Recall the notation of preceding Sections. Let $P \in C$ be a fixed smooth point. For each $i \in \{1, \dots, p\}$, let $Q_i \in C_i$ be a smooth point. For a line bundle L on C consider the sheaf

$$L_i := m_{Q_i} \otimes L.$$

According to [CoEP, p. 12-13] and [E01, Lemmas 30 and 31, p. 3068], there exists a unique twister $T_i := \mathcal{O}_C(Z_i)$ where Z_i is a formal sum of components of C such that $L_i \otimes T_i$ is P -quasistable.

For $i \in \{1, \dots, p\}$, set $\tilde{T}_i := \mathcal{O}_{\widetilde{C^2}}(\widetilde{C_i \times Z_i})$ and $\tilde{\mathcal{T}} = \bigotimes_{i=1}^n \tilde{T}_i$. If ϕ is a good partial desingularization, then $\widetilde{C_i \times Z_i}$ is a Cartier divisor of $\widetilde{C^2}$ for all $i \in \{1, \dots, p\}$. So, $\tilde{\mathcal{T}}$ is an invertible sheaf of $\widetilde{C^2}$.

Recall the definition of the sheaf on $\widetilde{C^2}/C$

$$\widetilde{\mathcal{M}} := \mathcal{I}_{\widetilde{\Delta}/\widetilde{C^2}} \otimes (p_2\phi)^* \mathcal{L} \otimes \tilde{\mathcal{T}},$$

where $\mathcal{I}_{\widetilde{\Delta}/\widetilde{C^2}}$ and $\tilde{\mathcal{T}}$ are invertible. We will show that $\phi_* \widetilde{\mathcal{M}}$ is a relatively torsion-free, rank-1 sheaf. For this we need the notion of admissibility introduced by [EP] and [CoEP].

Let $\gamma : \mathcal{X} \rightarrow \mathcal{S}$ be a family of connected curves and $\psi : \mathcal{Y} \rightarrow \mathcal{X}$ be a proper morphism such that the composition $\theta := \gamma \circ \psi$ is another family of curves. We say that ψ is a *semistable modification* of γ if for each geometric point $s \in \mathcal{S}$, there are a collection of nodes \mathcal{N}_s of the fiber \mathcal{X}_s and a map $\eta_s : \mathcal{N}_s \rightarrow \mathbb{N}$ such that the induced map $\psi_s : \mathcal{Y}_s \rightarrow \mathcal{X}_s$, where \mathcal{Y}_s is the fiber of θ over s , is \mathcal{X}_s -isomorphic to $\mu_{\eta_s} : (\mathcal{X}_s)_{\eta_s} \rightarrow \mathcal{X}_s$. If η_s is constant and equal to 1 for every s , we say that ψ is a *small semistable modification* of γ .

Assume ψ is a semistable modification of γ . Let \mathcal{L} be an invertible sheaf on \mathcal{Y} . We say that \mathcal{L} is ψ -admissible (resp. *negatively* ψ -admissible, resp. *positively* ψ -admissible, resp. ψ -invertible) at a given geometric point $s \in \mathcal{S}$ if the restriction of \mathcal{L} to every chain of rational curves of \mathcal{Y}_s over a node of \mathcal{X}_s has degree -1 , 0 or 1 . (resp. -1 or 0 , resp. 0 or 1 , resp. 0). We say that \mathcal{L} is ψ -admissible (resp. *negatively* ψ -admissible, resp. *positively* ψ -admissible, resp. ψ -invertible) if \mathcal{L} is so at every $s \in \mathcal{S}$. Notice that, if \mathcal{L} is negatively (resp. positively) ψ -admissible, for every chain of rational curves of \mathcal{Y}_s over a node of \mathcal{X}_s , the degree of \mathcal{L} on each component of the chain is 0 but for at most one component where the degree is -1 (resp. 1).

Proposition 4 *Let $\gamma : \mathcal{X} \rightarrow \mathcal{S}$ be a family of connected curves, $\psi : \mathcal{Y} \rightarrow \mathcal{X}$ a semistable modification of γ and $\theta := \gamma \circ \psi$. Let \mathcal{L} be an invertible sheaf on \mathcal{Y} of relative degree d over \mathcal{S} . Then the following statements hold:*

- 1) *The points $s \in \mathcal{S}$ at which \mathcal{L} is ψ -admissible (resp. negatively ψ -admissible, resp. positively ψ -admissible, resp. ψ -invertible) form an open set of \mathcal{S} .*
- 2) *\mathcal{L} is ψ -admissible if and only if $\psi_* \mathcal{L}$ is a relatively torsion-free, rank-1 sheaf on \mathcal{X}/\mathcal{S} of relative degree d , whose formation commutes with base change. In this case, $R^1 \psi_* \mathcal{L} = 0$.*
- 3) *If \mathcal{L} is ψ -admissible then the evaluation map $v : \psi^* \psi_* \mathcal{L} \rightarrow \mathcal{L}$ is surjective if and only if \mathcal{L} is positively ψ -admissible. Furthermore, v is bijective if and only if \mathcal{L} is ψ -invertible, if and only if $\psi_* \mathcal{L}$ is invertible.*

Proof. [EP, Theorem 3.1, p.9] ■

Proposition 5 *Let X be a curve and $\psi : Y \rightarrow X$ a semistable modification of X . Let P be a single point of Y not lying on any component contracted by ψ . Let \mathcal{E} be a locally free sheaf on X and \mathcal{L} an invertible sheaf on Y . Then \mathcal{L} is semistable (resp. P -quasistable, resp. stable) with respect to $\psi^*\mathcal{E}$ if and only if \mathcal{L} is ψ -admissible (resp. negatively ψ -admissible, resp. ψ -invertible) and $\psi_*\mathcal{L}$ is semistable (resp. $\psi(P)$ -quasistable, resp. stable) with respect to \mathcal{E} .*

Proof. [EP, Theorem 4.1, p.16] ■

According to Propositions 4 and 5, to conclude that $\phi_*\widetilde{\mathcal{M}}$ is a rank-1, torsion-free sheaf, it suffices to show that $\deg_E \widetilde{\mathcal{M}} \in \{-1, 0, 1\}$ for all ϕ -exceptional component $E \subset \widehat{C}$.

Notice that $(p_2\phi)^*\mathcal{L}|_{\widehat{C}}$ has degree 0 on a ϕ -exceptional component E . In the next lemma we calculate the degree of $\mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}}$ on each exceptional component $E \subset \widehat{C}$.

Lemma 6 *Let $\phi : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}^2$ be a good partial desingularization of \mathcal{C}^2 . If E is a ϕ -exceptional component such that $\phi(E) = (R, R)$, where R is a node of C , then $\deg_E(\mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}}) = 1$.*

Proof. By definition of good partial desingularization, the exceptional component E is obtained after blowing-up the diagonal subscheme Δ . Suppose $R \in C_i \cap C_j$. By Proposition 2, $\widetilde{\Delta}$ contains E and, equivalently, $E \subset \widetilde{C_i \times C_j}$. The figure 5 illustrates this situation:

We can see that C_i degenerates to \widehat{C}_i on the left side and degenerates to $\widehat{C}_i \cup E$ on the right side. Using the degeneration of C_i to \widehat{C}_i and $\deg_{C_i}(\mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}}) = -1$, it follows that

$$-1 = \deg_{C_i}(\mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}}) = \deg_{\widehat{C}_i}(\mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}}).$$

On the other hand, using the degeneration of C_i to $\widehat{C}_i \cup E$ we get

$$0 = \deg_{C_i}(\mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}}) = \deg_{\widehat{C}_i \cup E}(\mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}}),$$

so, $\deg_E(\mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}}) = 1$. The proof is complete. ■

To complete the analysis of the degree of the invertible sheaf $\widetilde{\mathcal{M}}$ over the ϕ -exceptional components contained in \widehat{C} , it remains to analyze the sheaf $\widetilde{\mathcal{T}}$.

Lemma 7 *Let E be a ϕ -exceptional component contained in the fiber $\widehat{C} = \rho^{-1}(R)$, where $R \in C_i \cap C_j$ is a node of C . Then,*

$$(a) \deg_E \left[\mathcal{O}_{\widetilde{\mathcal{C}^2}} \left(\widetilde{C_i \times Z_i} + \widetilde{C_j \times Z_i} \right) \right] = 0;$$

$$(b) \deg_E \left[\mathcal{O}_{\widetilde{\mathcal{C}^2}} \left(\widetilde{C_i \times Z_i} + \widetilde{C_j \times Z_j} \right) \right] = \deg_E \left[\mathcal{O}_{\widetilde{\mathcal{C}^2}} \left(\widetilde{C_j \times Z_j} - \widetilde{C_j \times Z_i} \right) \right].$$

Proof. (a) Assume that E is such that $\phi(E) = (R, S)$, with $S \in C_k \cap C_l$. Suppose that $Z_i = a_l C_l + a_k C_k + \dots$, where $a_l, a_k \in \mathbb{Z}$. Without loss generality, suppose that E is contained in the strict transform of $C_i \times C_l$ and that E is not

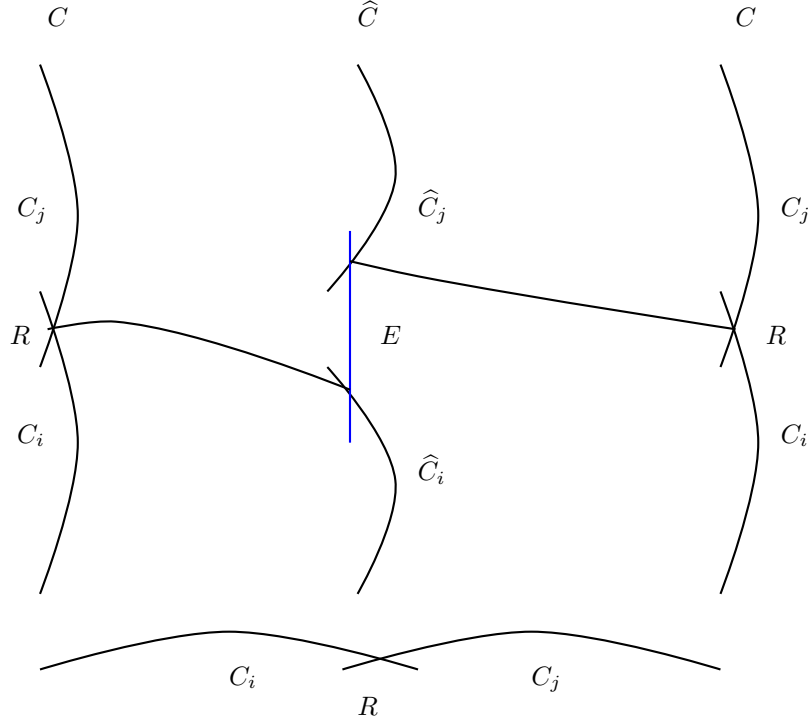


Figure 5: Local blowup.

contained in the strict transform of $C_s \times C_l$. According to Proposition 3, locally around E , we have

$$\mathcal{O}_{\widetilde{\mathcal{C}^2}}(\widetilde{C_i \times Z_i})|_{\widehat{\mathcal{C}}} = \mathcal{O}_{\widehat{\mathcal{C}}}(\cdots + a_l \widehat{C_l} + a_l E + a_k \widehat{C_k} + \cdots)$$

and

$$\mathcal{O}_{\widetilde{\mathcal{C}^2}}(\widetilde{C_j \times Z_i})|_{\widehat{\mathcal{C}}} = \mathcal{O}_{\widehat{\mathcal{C}}}(\cdots + a_l \widehat{C_l} + a_k E + a_k \widehat{C_k} + \cdots).$$

So

$$\deg_E \left[\mathcal{O}_{\widetilde{\mathcal{C}^2}}(\widetilde{C_i \times Z_i}) \right] = a_k - a_l$$

and

$$\deg_E \left[\mathcal{O}_{\widetilde{\mathcal{C}^2}}(\widetilde{C_j \times Z_i}) \right] = a_l - a_k.$$

Hence

$$\deg_E \left[\mathcal{O}_{\widetilde{\mathcal{C}^2}}(\widetilde{C_i \times Z_i} + \widetilde{C_j \times Z_i}) \right] = 0.$$

(b) Since

$$\widetilde{C_i \times Z_i} + \widetilde{C_j \times Z_j} = (\widetilde{C_i \times Z_i} + \widetilde{C_j \times Z_i}) + (\widetilde{C_j \times Z_j} - \widetilde{C_j \times Z_i}),$$

the result follows from (a). \blacksquare

The line bundle $\mathcal{O}_{\widetilde{\mathcal{C}^2}}(\widetilde{C_j \times Z_j} - \widetilde{C_j \times Z_i})$ over $\widetilde{\mathcal{C}^2}$ is called the *twister difference* with respect to i and j and it will denoted by \widetilde{T}_{j-i}

4 Twister difference

Let C be a connected, nodal curve with components C_1, \dots, C_p . Let $P \in C$ be a fixed smooth point.

Recall from Section 2 if I is an invertible sheaf on C , E is a polarization on C and Y is a subcurve of C , then

$$\beta_Y(I) = \chi(I_Y) + \frac{\deg_Y(E)}{\text{rk}(E)}.$$

Lemma 8 *Let I be an invertible sheaf on C and let E be a polarization on C . If Y and Z are subcurves of C such that $\dim Y \cap Z = 0$ or $Y \cap Z = \emptyset$, then*

$$\beta_{Y \cup Z}(I) = \beta_Y(I) + \beta_Z(I) - \delta_{YZ},$$

where $\delta_{YZ} = \#(Y \cap Z)$.

Proof. The proof when $Y \cap Z = \emptyset$ is trivial. If $\dim Y \cap Z = 0$, it follows from the exact sequence

$$0 \rightarrow I_Z(-Y \cap Z) \rightarrow I_{Y \cup Z} \rightarrow I_Y \rightarrow 0$$

that $\chi(I_{Y \cup Z}) = \chi(I_Y) + \chi(I_Z) - \delta_{YZ}$. As E is an invertible sheaf, it follows that

$$\frac{\deg_{Y \cup Z}(E)}{\text{rk}(E)} = \frac{\deg_Y(E)}{\text{rk}(E)} + \frac{\deg_Z(E)}{\text{rk}(E)}$$

and so

$$\begin{aligned} \beta_{Y \cup Z}(I) &= \chi(I_Y) + \chi(I_Z) - \delta_{YZ} + \frac{\deg_Y(E)}{\text{rk}(E)} + \frac{\deg_Z(E)}{\text{rk}(E)} \\ &= \beta_Y(I) + \beta_Z(I) - \delta_{YZ} \end{aligned}$$

and the proof is complete. ■

Let C_i and C_j be irreducible components of C such that $\delta_{i,j} > 0$. Consider $Q_i \in C_i$, $Q_j \in C_j$ and $P \in C_1$ nonsingular points of C . Let $T_i = \mathcal{O}_C(Z_i)|_C$ and $T_j = \mathcal{O}_C(Z_j)|_C$ be the twistors such that

$$M_i := m_{Q_i} \otimes L \otimes T_i \quad \text{and} \quad M_j := m_{Q_j} \otimes L \otimes T_j$$

are P -quasistable with respect to E .

Consider

$$M := m_{Q_j} \otimes L \otimes T_i.$$

If Y is a subcurve of C , we have three possibilities:

$$\beta_Y(M) = \beta_Y(M_i) - 1, \text{ if } Y \supset C_j \text{ and } Y \not\supset C_i \quad (3)$$

$$\beta_Y(M) = \beta_Y(M_i) + 1, \text{ if } Y \supset C_i \text{ and } Y \not\supset C_j \quad (4)$$

$$\beta_Y(M) = \beta_Y(M_i), \text{ otherwise.} \quad (5)$$

Let A and B be the following sets of subcurves of C :

$$\begin{aligned} A &= \{Y; \beta_Y(M) < 0 \text{ and } \beta_Y(M) \leq \beta_X(M), \forall X \text{ subcurve of } C\}; \\ B &= \{Y; \beta_Y(M) = 0, Y \supseteq C_1 \text{ and } \beta_Y(M) \leq \beta_X(M), \forall X \text{ subcurve of } C\}. \end{aligned}$$

Note that if A and B are empty, then M is P -quasistable. We now define a subcurve $Z_{i,j}$ of C in the following way. If $A \neq \emptyset$, then let $Z_{i,j}$ be a subcurve of A with minimal number of components. If $A = \emptyset$ and $B \neq \emptyset$, we let $Z_{i,j}$ be a subcurve of B with minimal number of component.

Notice that, in all situations, $Z_{i,j}$ contains C_j and does not contain C_i .

Proposition 9 *With the notation fixed above, the sheaf $M \otimes \mathcal{O}_C(-Z_{i,j})$ is P -quasistable.*

Proof. We consider only the case $A \neq \emptyset$. The case $A = \emptyset$ and $B \neq \emptyset$ is similar. Define $M' = M \otimes \mathcal{O}_C(-Z_{i,j})$.

Let Y be a subcurve of C . We can write

$$Y = Y_1 \cup Z_1,$$

with $Z_1 = \overline{Z_{i,j} - Y}$ and $Y_1 = \overline{Y - Z_{i,j}}$. Note that, by the minimality of $\beta_{Z_{i,j}}(M)$ and by Lemma 8, we have

$$\beta_{Z_{i,j}}(M) \leq \beta_{Z_{i,j} \cup Y_1}(M) = \beta_{Z_{i,j}}(M) + \beta_{Y_1}(M) - \delta_{Y_1, Z_{i,j}}.$$

So,

$$\beta_{Y_1}(M) - \delta_{Y_1, Z_{i,j}} \geq 0. \quad (6)$$

Now,

$$\beta_Y(M') = \beta_Y(M) + \delta_{Z_1 Z'_{i,j}} - \delta_{Y_1, Z_{i,j}} \quad (7)$$

$$= \beta_{Z_1}(M) + \beta_{Y_1}(M) - \delta_{Y_1 Z_1} + \delta_{Z_1 Z'_{i,j}} - \delta_{Y_1, Z_{i,j}} \quad (8)$$

$$= (\beta_{Y_1}(M) - \delta_{Y_1 Z_{i,j}}) + (\beta_{Z_1}(M) + \delta_{Z_1 Z'_{i,j}} - \delta_{Y_1 Z_1}) \quad (9)$$

$$\geq (\beta_{Y_1}(M) - \delta_{Y_1 Z_{i,j}}) + (\beta_{Z_{i,j}}(M) + \delta_{Z_1 Z'_{i,j}} - \delta_{Y_1 Z_1}), \quad (10)$$

where inequality (10) follows from the minimal property of $\beta_{Z_{i,j}}(M)$. Note that, by (6), the first parenthesis in (10) is nonnegative.

We have two cases to consider.

(I) C_i is not a component of Y .

In this case, since $C_j \subset Z_{i,j}$, we have $C_i \cap Z_{i,j} \neq \emptyset$ and so,

$$\delta_{Z_1 Z'_{i,j}} - \delta_{Y_1 Z_1} \geq 1.$$

By (3), we have $\beta_{Z_{i,j}}(M) \geq -1$ and we conclude that the second parenthesis in (10) is non negative too. Thus $\beta_Y(M') \geq 0$.

Now, if $Y \supset C_1$ and $C_1 \subset Z_{i,j}$ then, by (3), $\beta_{Z_{i,j}}(M) > -1$, so, the second parenthesis in (10) is positive and we conclude that $\beta_Y(M') \geq 0$. If $Y \supset C_1$, $C_1 \not\subset Z_{i,j}$ and $\beta_{Z_{i,j}}(M) > -1$ then, we are done, as above. If $Y \supset C_1$, $C_1 \not\subset Z_{i,j}$ and $\beta_{Z_{i,j}}(M) = -1$ then, since

$$C_1 \subseteq Z_{i,j} \cup Y_1,$$

it follows from (3) we have $\beta_{Z_{i,j} \cup Y_1} > -1$ and consequently (6) is strict. So, the first parenthesis in (10) is also strict. Thus, $\beta_Y(M') \geq 0$ in any case.

(II) C_i is a component of Y .

In this case, the subcurve Y_1 is nonempty and the subcurve $Z_{i,j} \cup Y_1$ contains C_i and C_j . Thus, by (5) we have

$$\beta_{Z_{i,j} \cup Y_1}(M) = \beta_{Z_{i,j}}(M) + \beta_{Y_1}(M) - \delta_{Y_1, Z_{i,j}} \geq 0$$

and, by (3), the inequality is strict if $Y_1 \supset C_1$.

Writing the right side of (10) in the form

$$(\beta_{Z_{i,j}}(M) + \beta_{Y_1}(M) - \delta_{Y_1, Z_{i,j}}) + (\delta_{Z_1 Z'_{i,j}} - \delta_{Y_1 Z_1})$$

we obtain, by (10), that

$$\beta_Y(M') \geq (\beta_{Z_{i,j}}(M) + \beta_{Y_1}(M) - \delta_{Y_1, Z_{i,j}}) + (\delta_{Z_1 Z'_{i,j}} - \delta_{Y_1 Z_1}). \quad (11)$$

Since the two parenthesis in (11) are non negative, we have $\beta_Y(M') \geq 0$. Moreover, if $C_1 \subset Y$ then, the first parenthesis in (11) is strict, so $\beta_Y(M') > 0$. Therefore the sheaf M' is P -quasistable and the proof is complete. ■

Corollary 10 *Keep the notation of Proposition 9. Then,*

$$T_j = T_i \otimes \mathcal{O}_C(-Z_{i,j}).$$

Proof. It follows from Proposition 9 and unicity of the twister. ■

Definition 11 *The subcurve $Z_{i,j}$ in the Proposition 9 is called the twister difference subcurve between j and i . We define $Z_{i,i} = \emptyset$.*

Let us go back to the analysis of $\tilde{\mathcal{T}}$. We have the following

Lemma 12 *Let E be a ϕ -exceptional component of \widehat{C} . Then,*

$$\deg_E \tilde{\mathcal{T}} = \deg_E (\widetilde{T_{j-i}}) = \deg_E \left[\mathcal{O}_{\widetilde{C^2}} \left(\widetilde{C_i \times Z_i} + \widetilde{C_j \times Z_j} \right) \right].$$

Proof. As we have

$$\deg_E \tilde{\mathcal{T}} = \deg_E \left[\mathcal{O}_{\widetilde{C^2}} \left(\widetilde{C_i \times Z_i} + \widetilde{C_j \times Z_j} \right) \right] + \deg_E \left[\mathcal{O}_{\widetilde{C^2}} \left(\sum_{r \neq i,j} \widetilde{C_r \times Z_r} \right) \right],$$

it is sufficient to prove that $\deg_E (\widetilde{C_r \times Z_r}) = 0$ if $r \neq i, j$. Suppose $E = \mu^{-1}(S)$ with $S \in C_k \cap C_l$ and $k, l \neq i, j$. According to Proposition 3, we have

$$\mathcal{O}_{\widetilde{C^2}} (\widetilde{C_r \times Z_r}) \Big|_{\widehat{C}} = \mathcal{O}_{\widehat{C}} (\cdots + 0 \cdot \widehat{C_k} + 0 \cdot F + 0 \cdot \widehat{C_l} + \cdots).$$

So, $\deg_E(\widetilde{C_r \times Z_r}) = 0$, if $r \neq i, j$. ■

Recall the sheaf $\widetilde{\mathcal{M}}$ defined in (1). Note that if \mathcal{E} is a polarization over $\mathcal{C} \rightarrow B$ then, the vector bundle $\widetilde{\mathcal{E}} = (p_1\phi)^*\mathcal{E}$ is a polarization on the family

$$\widetilde{\mathcal{C}^2} \xrightarrow{\phi} \mathcal{C}^2 \xrightarrow{p_1} \mathcal{C}.$$

So, we have $\deg_E(\widetilde{\mathcal{E}}) = 0$ and $\beta_E(\widetilde{\mathcal{M}}) = \deg(\widetilde{\mathcal{M}}) + 1$ for every ϕ -exceptional component E of $\widetilde{\mathcal{C}}$.

Proposition 13 *Let E be a ϕ -exceptional component of $\widehat{\mathcal{C}}$ with $R \in C_i \cap C_j$ and $E = \phi^{-1}(R, R)$. The following properties hold*

(i) *if $\mathcal{O}_C(Z_{i,j}) \simeq \mathcal{O}_C$ then, $\deg_E \widetilde{\mathcal{M}} = 1$ and $\beta_E(\widetilde{\mathcal{M}}) = 2$;*

(ii) *otherwise $\deg_E \widetilde{\mathcal{M}} = 0$ and $\beta_E(\widetilde{\mathcal{M}}) = 1$*

Proof. (i) Since $\mathcal{O}_C(Z_{i,j}) \simeq \mathcal{O}_C$, we have $\deg_E \widetilde{\mathcal{T}} = \deg_E \widetilde{T}_{j-i} = 0$ and hence $\deg_E \widetilde{\mathcal{M}} = \deg_E \mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}}$. By Lemma 6, we have $\deg_E \mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}} = 1$, from which we get

$$\deg_E \widetilde{\mathcal{M}} = 1 \text{ and } \beta_E(\widetilde{\mathcal{M}}) = 2$$

(ii) In this case the subcurve $Z_{i,j}$ contains C_j and does not contain C_i . By Proposition 2, we have

$$E \subset \widetilde{C_i \times C_j} \text{ and } E \subset \widetilde{C_j \times C_i}.$$

So, by Proposition 3,

$$\mathcal{O}_{\widetilde{\mathcal{C}^2}}(-\widetilde{C_j \times Z_{i,j}}) \Big|_{\widetilde{\mathcal{C}}} = \mathcal{O}_{\widehat{\mathcal{C}}}(\cdots 0 \cdot \widehat{C_i} + 0 \cdot E - 1 \cdot \widehat{C_j} + \cdots).$$

Hence

$$\deg_E \widetilde{\mathcal{M}} = \deg_E \mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}} + \deg_E \widetilde{\mathcal{T}} = 1 + (-1) = 0$$

and $\beta_F(\mathcal{M}) = 1$. ■

Proposition 14 *Let E be a ϕ -exceptional component of $\widehat{\mathcal{C}}$. Suppose $R \in C_i \cap C_j$ and $E = \phi^{-1}(R, S)$ with $S \in C_k \cap C_l$ and $S \neq R$. The following properties hold*

(i) *if $\mathcal{O}_C(Z_{i,j}) \simeq \mathcal{O}_C$ then, $\deg_E \widetilde{\mathcal{M}} = 0$ and $\beta_E(\widetilde{\mathcal{M}}) = 1$;*

(ii) *if $\mathcal{O}_C(Z_{i,j})$ is not trivial then, the following properties hold:*

(a) *If C_k and C_l are components of $Z_{i,j}$, then $\deg_E \widetilde{\mathcal{M}} = 0$ and $\beta_E(\widetilde{\mathcal{M}}) = 1$;*

(b) *If C_k and C_l are components of $Z'_{i,j}$, then $\deg_E \widetilde{\mathcal{M}} = 0$ and $\beta_E(\widetilde{\mathcal{M}}) = 1$;*

(c) *If $C_k \subset Z_{i,j}$ and $C_l \subset Z'_{i,j}$, then $\deg_E \widetilde{\mathcal{M}} \in \{-1, 1\}$ and $\beta_E(\widetilde{\mathcal{M}}) \in \{0, 2\}$*

Proof. Note that $\deg_E \mathcal{I}_{\widehat{\Delta}/\widehat{C}^2} = 0$. So, in this case, we have

$$\deg_E \widetilde{\mathcal{M}} = \deg_E \widetilde{\mathcal{T}} = \deg_E \widetilde{T}_{j-i}.$$

(i) If $\mathcal{O}_C(Z_{i,j}) \simeq \mathcal{O}_C$, then $\widetilde{\mathcal{T}} = \mathcal{O}_{\widehat{C}^2}$. So, $\deg_E \widetilde{\mathcal{M}} = 0$ and $\beta_E(\widetilde{\mathcal{M}}) = 1$.

(ii) (a) According to Proposition 3, we have two possibilities for $\mathcal{O}_{\widehat{C}^2}(-\widetilde{C_j \times Z_{i,j}})|_{\widehat{C}}$.

If $E \subset \widetilde{C_i \times C_l}$ and $E \subset \widetilde{C_j \times C_k}$ then,

$$\mathcal{O}_{\widehat{C}^2}(-\widetilde{C_j \times Z_{i,j}})|_{\widehat{C}} = \mathcal{O}_{\widehat{C}}(\cdots - 1 \cdot \widehat{C_k} - 1 \cdot E - 1 \cdot \widehat{C_l} + \cdots).$$

So, $\deg_E \widetilde{\mathcal{M}} = 0$ and $\beta_E(\widetilde{\mathcal{M}}) = 1$.

If $E \subset \widetilde{C_i \times C_k}$ and $E \subset \widetilde{C_j \times C_l}$ then,

$$\mathcal{O}_{\widehat{C}^2}(-\widetilde{C_j \times Z_{i,j}})|_{\widehat{C}} = \mathcal{O}_{\widehat{C}}(\cdots - 1 \cdot \widehat{C_k} - 1 \cdot E - 1 \cdot \widehat{C_l} + \cdots).$$

So, $\deg_E \widetilde{\mathcal{M}} = 0$ and $\beta_E(\widetilde{\mathcal{M}}) = 1$.

(b) In this case we have

$$\mathcal{O}_{\widehat{C}^2}(-\widetilde{C_j \times Z_{i,j}})|_{\widehat{C}} = \mathcal{O}_{\widehat{C}}(\cdots + 0 \cdot \widehat{C_k} + 0 \cdot E + 0 \cdot \widehat{C_l} + \cdots).$$

So, $\deg_E \widetilde{\mathcal{M}} = 0$ and $\beta_E(\widetilde{\mathcal{M}}) = 1$.

(c) In this case, we have two possibilities for $\mathcal{O}_{\widehat{C}^2}(-\widetilde{Z_{i,j} \times C_j})|_{\widehat{C}}$.

If $E \subset \widetilde{C_i \times C_k}$ and $E \subset \widetilde{C_j \times C_l}$ then,

$$\mathcal{O}_{\widehat{C}^2}(-\widetilde{C_j \times Z_{i,j}})|_{\widehat{C}} = \mathcal{O}_{\widehat{C}}(\cdots - 1 \cdot \widehat{C_k} + 0 \cdot E + 0 \cdot \widehat{C_l} + \cdots).$$

So, $\deg_E \widetilde{\mathcal{M}} = -1$ and $\beta_E(\widetilde{\mathcal{M}}) = 0$.

If $E \subset \widetilde{C_i \times C_l}$ and $E \subset \widetilde{C_j \times C_k}$ then,

$$\mathcal{O}_{\widehat{C}^2}(-\widetilde{C_j \times Z_{i,j}})|_{\widehat{C}} = \mathcal{O}_{\widehat{C}}(\cdots - 1 \cdot \widehat{C_k} + -1 \cdot E + 0 \cdot \widehat{C_l} + \cdots).$$

So, $\deg_E \widetilde{\mathcal{M}} = 1$ and $\beta_E(\widetilde{\mathcal{M}}) = 2$.

Thus, the proof is complete. ■

Proposition 15 *Keep the notation of Proposition 14 and suppose that $\mathcal{O}_C(Z_{i,j})$ is not trivial. Let Y be a subcurve of C and let E be a ϕ -exceptional component of \widehat{C} . Suppose $E = \phi^{-1}(R, S)$, with $S \in Y \cap Y'$ and $R \neq S$.*

(a) *If $Y \subset Z_{i,j}$ and $E \subset \widetilde{C_i \times Y}$, then $\beta_E(\widetilde{\mathcal{M}}) \in \{0, 1\}$;*

(b) *If $Y \subset Z_{i,j}$ and $E \subset \widetilde{C_j \times Y}$, then $\beta_E(\widetilde{\mathcal{M}}) \in \{1, 2\}$;*

(c) *If $Y \subset Z'_{i,j}$ and $E \subset \widetilde{C_i \times Y}$, then $\beta_E(\widetilde{\mathcal{M}}) \in \{1, 2\}$;*

(d) *If $Y \subset Z'_{i,j}$ and $E \subset \widetilde{C_j \times Y}$, then $\beta_E(\widetilde{\mathcal{M}}) \in \{0, 1\}$.*

Proof. Let C_k and C_l be irreducibles components of C such that $C_k \subset Y$, $C_l \subset Y'$ and $S \in C_k \cap C_l$.

(a) If C_k and C_l are components of $Z_{i,j}$, then, by Proposition 14(ii)(a), $\beta_E(\widetilde{\mathcal{M}}) = 1$. If C_k and C_l are components of $Z_{i,j}$ and $Z'_{i,j}$, respectively, then, by hypothesis, $E \subset \widetilde{C_i \times C_k}$ and according to Proposition 14(ii)(c), we have $\beta_E(\widetilde{\mathcal{M}}) = 0$.

(b) If C_k and C_l are components of $Z_{i,j}$ then, by Proposition 14(ii)(a), $\beta_E(\widetilde{\mathcal{M}}) = 1$. If C_k and C_l are components of $Z_{i,j}$ and $Z'_{i,j}$, respectively, then, by hypothesis, $E \subset \widetilde{C_j \times C_k}$ and according to Proposition 14(ii)(c), we have $\beta_E(\widetilde{\mathcal{M}}) = 2$.

The cases (c) and (d) are similar and the proof is complete. \blacksquare

Corollary 16 *The sheaf $\phi_* \widetilde{\mathcal{M}}$ is relatively torsion-free and rank-1 and its formation commutes with base changes.*

Proof. By Proposition 4, we need to show that $\widetilde{\mathcal{M}}$ is ϕ -admissible and this follows from Propositions 13 and 14. \blacksquare

5 Degree-1 Abel maps

Recall some notation introduced in the previous sections. Let $\phi : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}^2$ be a good partial desingularization of \mathcal{C}^2 , $\mu := (p_2 \circ \phi)|_{\widehat{C}} : \widehat{C} \rightarrow C$, and $\rho = p_1 \circ \phi$, where p_1 and p_2 are, respectively, the projections of \mathcal{C}^2 onto the first and second factors.

Let $R \in C_i \cap C_j$ be a fixed reduced node of C . We know that $\widehat{C} = \rho^{-1}(R) = \mu^{-1}(C)$. Let $\Delta \subset \mathcal{C}^2$ be the diagonal subscheme and let $\widetilde{\Delta}$ be the strict transform of Δ (via ϕ).

We will prove that the sheaf $\phi_* \widetilde{\mathcal{M}}$ is σ -quasistable, where

$$\widetilde{\mathcal{M}} := \mathcal{I}_{\widetilde{\Delta}/\widetilde{\mathcal{C}^2}} \otimes (p_2 \phi)^* \mathcal{L} \otimes \widetilde{\mathcal{T}}, \quad (12)$$

is a sheaf over $\widetilde{\mathcal{C}^2}/\mathcal{C}$ with \mathcal{L} is a line bundle over \mathcal{C}/B , $\widetilde{\mathcal{T}} = \bigotimes_{i=1}^p \widetilde{T}_i$ and T_i is the twister for each component C_i of C .

Let $\sigma : \mathcal{C} \rightarrow \mathcal{C}^2$ be a section through the smooth locus of $f : \mathcal{C} \rightarrow B$ such that $\sigma(0) = P$. Let φ be the restriction of ϕ to the inverse image of the smooth locus of \mathcal{C}^2 . Since φ is an isomorphism, there exists a lifting of σ to $\widetilde{\mathcal{C}^2}$, which we also denote by σ . We will denote $\widehat{P} := \phi^{-1}(P)$, which is a smooth point of \widehat{C} .

A key tool to prove σ -quasistability of $\widetilde{\mathcal{M}}$ is the following result.

Proposition 17 *Let $\psi : \mathcal{Y} \rightarrow \mathcal{C}^2$ be a good partial desingularization of \mathcal{C}^2 . Let \mathcal{L} and \mathcal{M} be ψ -admissible invertible sheaves on \mathcal{Y} . Let Y and X be fibers, respectively of*

$$\mathcal{Y} \xrightarrow{\psi} \mathcal{C}^2 \xrightarrow{p_i} \mathcal{C}$$

and of

$$\mathcal{C}^2 \xrightarrow{p_i} \mathcal{C}$$

such that $Y = \psi^{-1}(X)$, where p_i is the projection onto the i -th factor. Let L and M be, respectively, the restrictions of \mathcal{L} and \mathcal{M} to Y and assume that

$$L \otimes M^{-1} = \mathcal{O}_Y \left(\sum_i a_i E_i \right),$$

where the sum runs over all $E_i \subset Y$ contracted by ψ and $a_i \in \mathbb{Z}$.

Then, $\psi_*(L) \cong \psi_*(M)$.

Proof. [EP, Proposition 3.2, p.13] ■

Recall that, by Proposition 3, $\widehat{C} = \rho^{-1}(R)$ consists of one of the following types: $\widehat{C} \cong C$ if R is a smooth point; $\widehat{C} \cong C_R$ if R is an irreducible node or $\widehat{C} \cong C(1)$ if R is a reducible node of C . In this way, each connected subcurve \widehat{Y} of \widehat{C} is of the form:

$$\widehat{Y} = A \cup B \cup D,$$

where,

- $A = \bigcup_{k=1}^r \widehat{C}_{i_k}$, with $\{i_k, \dots, i_r\} \subset \{1, \dots, p\}$ and $\mu(\widehat{C}_{i_k}) \cong C_{i_k}$ with C_{i_k} an irreducible component of C ;
- $B = \bigcup E_{i_l}$, with E_{i_l} a smooth rational component which is equal to $\mu^{-1}(R)$ for some node R of $Y = \bigcup_{k=1}^r C_{i_k}$;
- $D = \bigcup E_{i_m}$, with E_{i_m} a smooth rational component which is equal to $\mu^{-1}(R)$ for some node $R \in Y \cap Y'$.

In this case, we say that \widehat{Y} is a Y -lifting. Note that each subcurve Y gives rise to more than one subcurve \widehat{Y} , however a given \widehat{Y} is the Y -lifting of exactly one subcurve Y of C .

Our goal is to prove that $\phi_* \widetilde{\mathcal{M}}$ is σ -quasistable. Since by Corollary 16 the formation of $\phi_* \widetilde{\mathcal{M}}$ commutes with base change, it suffices to show that the sheaf $\phi_* \left(\widetilde{\mathcal{M}}|_{\widehat{C}} \right)$ is \widehat{P} -quasistable for every fiber \widehat{C} of $\phi : \widetilde{\mathcal{C}^2} \rightarrow \mathcal{C}$.

Fix a fiber \widehat{C} of ϕ . We define the sheaf

$$\mathcal{G} := \widetilde{\mathcal{M}}|_{\widehat{C}} \left(\sum_i E_i \right),$$

where the sum is taken over all ϕ -exceptional component E_i of \widehat{C} such that $\beta_{E_i}(\widetilde{\mathcal{M}}) = 2$.

Proposition 18 *The sheaf \mathcal{G} is \widehat{P} -quasistable.*

Proof. If \widehat{C} is a fiber over a smooth point $R \in C_i$ then, by Proposition 3, $\widehat{C} \cong C$. In this case $\mathcal{G} \cong \widetilde{\mathcal{M}}|_{\widehat{C}}$ and $\widetilde{\mathcal{T}} \cong \mathcal{O}_{\widehat{C}^2}(C_i \times \widetilde{Z_{i,i}})$. Therefore,

$$\mathcal{G} \cong \widetilde{\mathcal{M}}|_{\widehat{C}} \cong m_R \otimes L \otimes T_i$$

which is \widehat{P} -quasistable.

We can assume that \widehat{C} is the fiber over a node of C . Let \widehat{Y} of \widehat{C} . We have to prove that $\beta_{\widehat{Y}}(\mathcal{G}) \geq 0$ and $\beta_{\widehat{Y}}(\mathcal{G}) > 0$ if $\widehat{P} \in \widehat{Y}$. We have to consider two cases.

(I) \widehat{C} is the fiber over an irreducible node R .

In this case, by Proposition 3, there is only one ϕ -exceptional component E contained in \widehat{C} . Suppose $R \in C_i$. We have three possibilities for a subcurve $\widehat{Y} \subset \widehat{C}$.

If $\widehat{Y} = E$ then, since R is an irreducible node, we have

$$\deg_E(\widetilde{\mathcal{T}}) = 0 \text{ and } \deg_E(\widetilde{\mathcal{M}}) = \deg_E(\mathcal{I}_{\widehat{\Delta}/\widehat{C}^2}) = 1.$$

It follows that $\beta_{\widehat{Y}}(\widetilde{\mathcal{M}}) = 2$. By definition of the sheaf \mathcal{G} , we have $\beta_{\widehat{Y}}(\mathcal{G}) = 0$.

If \widehat{Y} is a Y -lifting and $Y \not\supset C_i$ then, $\widehat{Y} \not\supset E$. Hence $\widehat{Y} \cong Y$ and

$$\beta_{\widehat{Y}}(\mathcal{G}) = \beta_Y(\widetilde{\mathcal{M}}) \geq 0.$$

If \widehat{Y} is a Y -lifting and $Y \supset C_i$ then, we have two subcases. First, if $E \subset \widehat{Y}$ then, by the flatness of ρ we have $\beta_{\widehat{Y}}(\mathcal{G}) = \beta_Y(\widetilde{\mathcal{M}}) \geq 0$. Second, if $E \not\subset \widehat{Y}$ then, the first case implies $\beta_{\widehat{Y} \cup E}(\mathcal{G}) \geq 0$. Since $\beta_E(\mathcal{G}) = 0$, it follows from Lemma 8 that $\beta_{\widehat{Y}}(\mathcal{G}) + \beta_E(\mathcal{G}) - 2 \geq 0$ and, therefore, $\beta_{\widehat{Y}}(\mathcal{G}) \geq 2$.

(II) \widehat{C} is the fiber over a reducible node $R \in C_i \cap C_j$.

In this case, by Proposition 3, $\widehat{C} \cong C(1)$ where $C(1)$ is obtained from C by replacing each reducible node by an ϕ -exceptional component.

If $\widehat{Y} = E$ with E a ϕ -exceptional component then, by construction of \mathcal{G} and by Propositions 13 and 14, we have $\beta_E(\widetilde{\mathcal{M}}) \in \{0, 1, 2\}$ and hence $\beta_E(\mathcal{G}) \in \{0, 1\}$.

If \widehat{Y} contains components which are not ϕ -exceptional then, let Y be the subcurve of C such that \widehat{Y} is a Y -lifting. Either $Y = C$ or Y is a proper subcurve of C .

If $Y = C$ then, we can write

$$\widehat{C} = \widehat{Y} \cup B_0 \cup B_1,$$

where B_0, B_1 are sets of ϕ -exceptional components of \widehat{C} such that $\beta_E(\mathcal{G}) = 0$ for every $E \in B_0$ and $\beta_E(\mathcal{G}) = 1$ for every $E \in B_1$. Notice that, since $Y = C$, by definition of the curve \widehat{C} , each ϕ -exceptional component contained either B_0 or B_1 intersects \widehat{C} in two points. So, by Lemma 8, we have

$$0 = \beta_{\widehat{C}}(\mathcal{G}) = \beta_{\widehat{C}}(\widetilde{\mathcal{M}}) - 2\#B_0 - \#B_1.$$

Since \widehat{Y} is proper, at least one of B_0 or B_1 is nonempty and so, $\beta_{\widehat{Y}}(\mathcal{G}) > 0$.

If Y is a proper subcurve of C then, we can write $Y = Y_1 \cup Y_2$, with $Y_1 \subset Z_{i,j}$ and $Y_2 \subset Z'_{i,j}$, where $Z_{i,j}$ is the twister difference subcurve with respect to i

and j (see Definition 11). Notice that Y_1 or Y_2 may be empty, that is, Y may be contained in $Z_{i,j}$ or in $Z'_{i,j}$. We can write $\widehat{Y} = \widehat{Y}_1 \cup \widehat{Y}_2$, where \widehat{Y}_k is a Y_k -lifting for $k = 1, 2$. Let

$$\widetilde{Y}_1 := \widetilde{C_i \times Y_1} \cap \widehat{C}$$

and

$$\widetilde{Y}_2 := \widetilde{C_j \times Y_2} \cap \widehat{C}.$$

Notice that, by construction, \widetilde{Y}_1 and \widetilde{Y}_2 contains all the ϕ -exceptional components of \widehat{C} contained, respectively, in $\widetilde{C_i \times Y_1}$ and $\widetilde{C_j \times Y_2}$. By Proposition 15 we see that each ϕ -exceptional component E contained in \widetilde{Y}_1 satisfies $\beta_E(\widetilde{\mathcal{M}}) \in \{0, 1\}$. So, by the flatness of ρ and by the definition of \mathcal{G} we have

$$\beta_{Y_1}(\widetilde{\mathcal{M}}) = \beta_{\widetilde{Y}_1}(\mathcal{G}). \quad (13)$$

Similarly we have

$$\beta_{Y_2}(\widetilde{\mathcal{M}}) = \beta_{\widetilde{Y}_2}(\mathcal{G}). \quad (14)$$

We need to prove the following two claims.

Claim 1 *Every ϕ -exceptional component E not contained in the region $\widetilde{C_i \times Y_1}$ and such that E intersects \widetilde{Y}_1 satisfies*

$$\beta_{\widetilde{Y_1 \cup E}}(\mathcal{G}) = \beta_{\widetilde{Y}_1}(\widetilde{\mathcal{M}}).$$

Indeed, by the definition of the sheaf \mathcal{G} , we have $\beta_E(\mathcal{G}) \in \{0, 1\}$, for every ϕ -exceptional component E . If E is such that $\beta_E(\mathcal{G}) = 1$ then, by Lemma 8 and by Equation (13), we have

$$\beta_{\widetilde{Y_1 \cup E}}(\mathcal{G}) = \beta_{\widetilde{Y}_1}(\mathcal{G}) + \beta_E(\mathcal{G}) - 1 = \beta_{\widetilde{Y}_1}(\mathcal{G}) = \beta_{\widetilde{Y}_1}(\widetilde{\mathcal{M}}).$$

If E is such that $\beta_E(\mathcal{G}) = 0$ then, since by hypothesis $E \not\subset \widetilde{C_i \times Y_1}$ and E intersects \widetilde{Y}_1 , it follows that $E \subset \widetilde{C_j \times Y_1}$. By definition of the sheaf \mathcal{G} , we have $\beta_E(\widetilde{\mathcal{M}}) = 2$. So, again by Lemma 8 and by Equation (13), we have

$$\beta_{\widetilde{Y_1 \cup E}}(\mathcal{G}) = \beta_{\widetilde{Y}_1}(\mathcal{G}) - 1 = \beta_{\widetilde{Y}_1}(\widetilde{\mathcal{M}}) + 1 - 1 = \beta_{\widetilde{Y}_1}(\widetilde{\mathcal{M}}).$$

In a similar way we can prove that

$$\beta_{\widetilde{Y_2 \cup E}}(\mathcal{G}) = \beta_{\widetilde{Y}_2}(\widetilde{\mathcal{M}}),$$

for every ϕ -exceptional component E not contained in $\widetilde{C_2 \times Y_2}$ and such that E intersects \widetilde{Y}_2 . \square

Claim 2 *For every Y_k -lifting \overline{Y}_k contained in \widetilde{Y}_k and $k = 1, 2$, we have*

$$\beta_{\overline{Y}_k}(\mathcal{G}) \geq \beta_{\widetilde{Y}_k}(\mathcal{G}).$$

Indeed, by the definition of the sheaf \mathcal{G} , we have $\beta_E(\mathcal{G}) \in \{0, 1\}$, for every ϕ -exceptional component E . If $\overline{Y}_i \subset \widetilde{Y}_i$ is a Y_i -lifting then,

$$\widetilde{Y}_i = \overline{Y}_i \cup A_1 \cup A_0,$$

where $A_1 \subset \widetilde{Y}_k$ is the set of ϕ -exceptional components contained in \widetilde{Y}_k , not contained in \overline{Y}_k and such that $\beta_E(\mathcal{G}) = 1$ for every $E \in A_1$ and $A_0 \subset \widetilde{Y}_k$ is the set of ϕ -exceptional components contained in \widetilde{Y}_k , not contained in \overline{Y}_k and such that $\beta_E(\mathcal{G}) = 0$ for every $E \in A_0$. By Lemma 8, we have

$$\begin{aligned} \beta_{\widetilde{Y}_k}(\mathcal{G}) &\geq \beta_{\overline{Y}_k}(\mathcal{G}) + \beta_{A_1}(\mathcal{G}) - \#(A_1 \cap \overline{Y}_k) + \beta_{A_0}(\mathcal{G}) - \#(A_0 \cap \overline{Y}_k) \\ &= \beta_{\overline{Y}_k}(\mathcal{G}) + \#A_1 - \#(A_1 \cap \overline{Y}_k) - \#(A_0 \cap \overline{Y}_k), \end{aligned}$$

where the above inequality follows from the fact that each ϕ -exceptional component contained in either A_1 or A_0 intersects \widehat{Y}_k in either 1 or 2 points. Hence, the integer

$$\#A_1 - \#(A_1 \cap \overline{Y}_k) - \#(A_0 \cap \overline{Y}_k)$$

in the above equation is nonpositive and we conclude that $\beta_{\overline{Y}_k}(\mathcal{G}) \geq \beta_{\widetilde{Y}_k}(\mathcal{G})$. \square

We can conclude the proof as follows. By Lemma 8 and by the fact that the sheaf $\widetilde{\mathcal{M}}$ is generically σ -quasistable, we have

$$\beta_{Y_1 \cup Y_2}(\widetilde{\mathcal{M}}) = \beta_{Y_1}(\widetilde{\mathcal{M}}) + \beta_{Y_2}(\widetilde{\mathcal{M}}) - \#(Y_1 \cap Y_2) \geq 0,$$

with strict inequality if $P \in Y_1 \cup Y_2$. By claims 1 and 2, to check that \mathcal{G} is $\widetilde{\sigma}$ -quasistable, we can reduce to check the condition for the case $\widehat{Y}_k = \widetilde{Y}_k$, $k = 1, 2$.

Let D_0 and D_1 be respectively the sets of ϕ -exceptional components of \widehat{C} contained in $\widetilde{Y}_1 \cap \widetilde{Y}_2$ and such that $\beta_E(\mathcal{G}) = 0$ for every $E \in D_0$ and $\beta_E(\mathcal{G}) = 1$ for every $E \in D_1$. We can write

$$\widetilde{Y}_1 = \overline{Y}_1 \cup D_0 \cup D_1 \quad \text{and} \quad \widetilde{Y}_2 = \overline{Y}_2 \cup D_0 \cup D_1,$$

where

$$\overline{Y}_k = \widetilde{Y}_k \setminus (D_0 \cup D_1), \tag{15}$$

for $k = 1, 2$. By Lemma 8 and by Equation (15), we have

$$\begin{aligned} \beta_{\widehat{Y}}(\mathcal{G}) &= \beta_{\widetilde{Y}_1 \cup \widetilde{Y}_2}(\mathcal{G}) \\ &= \beta_{\overline{Y}_1 \cup \overline{Y}_2}(\mathcal{G}) \\ &= \beta_{\overline{Y}_1}(\mathcal{G}) + \beta_{\overline{Y}_2}(\mathcal{G}) - \#(\overline{Y}_1 \cap \overline{Y}_2) \\ &\geq \beta_{\widetilde{Y}_1}(\mathcal{G}) + \beta_{\widetilde{Y}_2}(\mathcal{G}) - \#(\widetilde{Y}_1 \cap \widetilde{Y}_2) \end{aligned} \tag{16}$$

$$\begin{aligned} &= \beta_{\widetilde{Y}_1}(\widetilde{\mathcal{M}}) + \beta_{\widetilde{Y}_2}(\widetilde{\mathcal{M}}) - \#(\widetilde{Y}_1 \cap \widetilde{Y}_2) \\ &= \beta_{Y_1}(\widetilde{\mathcal{M}}) + \beta_{Y_2}(\widetilde{\mathcal{M}}) - \#(\widetilde{Y}_1 \cap \widetilde{Y}_2) \end{aligned} \tag{17}$$

$$\geq \beta_{Y_1}(\widetilde{\mathcal{M}}) + \beta_{Y_2}(\widetilde{\mathcal{M}}) - \#(Y_1 \cap Y_2) \tag{18}$$

$$\geq 0 \tag{19}$$

where Inequality (16) follows from Claim 2, Equation (17) follows from the flatness of the sheaf $\widetilde{\mathcal{M}}$ and Inequality (18) follows from the fact that $\#(Y_1 \cap Y_2) \geq \#D_0 + \#D_1$. Note that if $\widehat{P} \in \widehat{Y}$ then Inequality (19) is strict.

The proof is complete. ■

Proposition 19 *The sheaf $\phi_*\widetilde{\mathcal{M}}$ is σ -quasistable.*

Proof. By Corollary 16 the sheaf $\phi_*\widetilde{\mathcal{M}}$ is torsion-free of rank-1. By Proposition 18, the sheaf \mathcal{G} is $\widetilde{\sigma}$ -quasistable, so by Proposition 4, we have that $\phi_*\mathcal{G}$ is σ -quasistable. By Proposition 17, we have $\phi_*\mathcal{G} \cong \phi_*\widetilde{\mathcal{M}}$. The proof is complete. ■

Theorem 20 (Main result) *The sheaf $\phi_*\widetilde{\mathcal{M}}$ on $p_1 : \mathcal{C}^2 \rightarrow \mathcal{C}$ induces a morphism*

$$\begin{aligned} \overline{\alpha}_{\mathcal{L},\varepsilon} : \mathcal{C} &\rightarrow \overline{\mathcal{J}}_{\varepsilon}^{\sigma} \\ Q &\mapsto \left[\phi_*\widetilde{\mathcal{M}} \Big|_{p_1^{-1}(Q)} \right] \end{aligned}$$

restricting to $\alpha_{\mathcal{L},\varepsilon}$ over the smooth locus of $f : \mathcal{C} \rightarrow B$.

Proof. By Corollary 16 the sheaf $\phi_*\widetilde{\mathcal{M}}$ is torsion-free of rank-1. By Proposition 19 $\phi_*\widetilde{\mathcal{M}}$ is σ -quasistable and hence $\overline{\alpha}_{\mathcal{L},\varepsilon}$ is a morphism. As ϕ is an isomorphism away from the exceptional components, it follows that $\overline{\alpha}_{\mathcal{L},\varepsilon}$ extend $\alpha_{\mathcal{L},\varepsilon}$. Thus, the proof is complete. ■

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